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Rational and Polynomial Interpolation of Analytic Functions with Restricted Growth

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Let f be an analytic function on a domain $D \subset \mathbb{C} \cup \{\infty\}$ and r_n the rational function of degree n with poles at the points $B_n = \{b_{ni}\}_{i=1}^n$, interpolating to f at the points $A_n = \{a_{ni}\}_{i=0}^n \subset D$. A fundamental question is whether it is possible to choose the points A_n and B_n so that r_n converges locally uniformly to f on D for every analytic function f on D. In some situations the interpolation points must be allowed to approach the boundary of D as n tends to infinity and then we cannot obtain convergence for every analytic f on D. If we restrict the growth of $f(z)$ when z goes to the boundary of D, we still have some positive convergence results that we prove here. © 2001 Academic Press

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1. INTRODUCTION

Consider a domain $D \subset \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and an analytic function f on D. Let for each $n \geqslant 1$, $A_n = \{a_{ni}\}_{i=0}^n \subset D$ and $B_n = \{b_{ni}\}_{i=1}^n \subset \overline{\mathbb{C}}$ be two collections of points (some a_{ni} or some b_{ni} may coincide). Then there exists a unique rational function r_n, of order n (the degree of the numerator and the denominator are at most n), with poles at B_n, interpolating to f at A_n, counting multiplicities (see [8, Section 8.1]). If A_n and B_n have some common points, we cancel them in the construction of r_n. The question of study in this paper is whether, given D, it is possible to choose poles B_n and interpolation points A_n in such a way that for all analytic f, r_n converges to f, uniformly on compact subsets of D, as n tends to infinity.

This type of problem has been studied by Walsh (see [8]) and Bagby (see [4]) and more recently by Ambroladze and Wallin in the papers [1], [2] and [3]. If we consider interpolating polynomials (which corresponds to the case where all poles are at infinity) on a bounded simply connected domain $D \subset \mathbb{C}$, and apply results from [1] and [3] we have the following.

Such interpolation points, guaranteeing convergence for all analytic functions on D exist if and only if the boundary of D is an analytic curve.

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We also have that these interpolation points must satisfy the following necessary and sufficient condition.

1. $\overline{\bigcup_{n\geqslant} A_n} \subset D$.

2. If α is a weak star limit for any subsequence of the point counting measures

$$\alpha_n = \frac{1}{n+1}\sum_{i=0}^{n} \delta_{a_{ni}},$$

then the sweeping out measure (to be defined in Section 2.1) α' of α onto ∂D is the equilibrium measure on ∂D.

The reason for this is as follows. If condition 1 above does not hold, then by [3, Theorem 5], there exists, for an arbitrary point $z_0 \in D \setminus \bigcup_{n\geqslant 1} A_n$ an analytic function f on D such that

$$\limsup_{n\to\infty} |f(z_0) - r_n(z_0)| = \infty.$$

If instead condition 1 above does hold, then by [1, Theorems 1 and 2], we get convergence for every analytic f on D if and only if condition 2 above also is true. Note that the sequence of measures α_n above always has a weak star convergent subsequence, and note also that given a unit measure α with $\operatorname{supp} \alpha \subset D$, there is, by Lemma 4.2 in this paper, a sequence of normalised point counting measures an converging to α in the weak star sense. We see that the kind of interpolation points that we seek, exist if and only if there is a measure α, with $\operatorname{supp} \alpha \subset D$ such that α' is the equilibrium measure on ∂D, but by [3, Theorem 1], such α exist if and only if ∂D is an analytic curve.

If we, instead of demanding convergence for *any* analytic function f on D, we consider only *bounded* analytic functions on a simply connected domain D, or analytic functions dominated by some locally integrable function, then Theorem 3.1 in this paper tells us that there are interpolation points guaranteeing convergence, no matter how nonsmooth the boundary may be. The crux is, as we saw in the reasoning above, that for non analytic boundaries we have to let the interpolation points approach the boundary, while for analytic boundaries, the interpolation points can be separated from the boundary. When we let the interpolation points approach the boundary, it is natural to expect (and formally proven in [3, Theorem 5]) that we cannot obtain convergence for any analytic function on D. Analytic functions can behave very wildly near the boundary and we may get "bad information" by interpolating at such points.

Returning to the more general case of rational interpolation, we have a similar situation. If we allow the interpolation points to approach the boundary of D, we cannot get convergence for all analytic functions on D (by the same theorem). There is however in [2, Theorem 1], a condition on the poles B_n and interpolation points A_n that guarantee convergence for every bounded analytic function on D, where the interpolation points are allowed to approach ∂D. The main theorem of this paper (Theorem 3.2) tells us that under rather general assumptions, we can find poles and interpolation points satisfying that condition.

2. DEFINITIONS AND NOTATION

If not explicitly stated otherwise, this is our notation:

$\overline{\mathbb{C}}$	The extended complex plane, $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.
E	A Borel subset of $\overline{\mathbb{C}}$.
D	A domain in $\overline{\mathbb{C}}$.
K	A compact set in $\overline{\mathbb{C}}$.
A_n, B_n	Sets of points in $\overline{\mathbb{C}}$, where $A_n = \{a_{ni}\}_{i=0}^{n}$, $B_n = \{b_{ni}\}_{i=1}^{n}$.
μ	A probability measure on $\overline{\mathbb{C}}$.
supp μ	The support of the measure μ.
δ_z	The Dirac measure at z.
α_n, β_n	The normalised point counting measures of A_n and B_n, $\alpha_n = \frac{1}{n+1}\sum_{i=0}^{n} \delta_{a_{ni}}$, $\beta_n = \frac{1}{n}\sum_{i=1}^{n} \delta_{b_{ni}}$.
$\mu_n \xrightarrow{w^*} \mu$	Weak star convergence of measures, $\int \phi \, d\mu_n \to \int \phi \, d\mu$ for every continuous function ϕ on $\overline{\mathbb{C}}$.
$U_\mu(z)$	The logarithmic potential of the measure μ, $U_\mu(z) = -\int \log \lvert z-t \rvert \, d\mu(t) \, d\mu(z)$.
$I(\mu)$	The energy of the measure μ, $I(\mu) = \int U_\mu(z) \, d\mu(z) = -\iint \log \lvert z-t \rvert \, d\mu(t) \, d\mu(z)$.
$g_D(t, z)$	The Green function of the domain D, ∂D non-polar, with pole at $z \in D$.
$\lVert \cdot \rVert_K$	The supremum norm over K.
$C(K)$	The space of continuous functions ϕ: $K \to \mathbb{R}$.

2.1. *Sweeping out of Measures*

We define the sweeping out of a measure using the solution to Dirichlet's problem and the Riesz representation theorem.

For a regular domain D in $\overline{\mathbb{C}}$, and a finite positive measure μ with supp $\mu \subset \bar{D}$, we can define a positive linear functional L on $C(\partial D)$ in the following way. For every $\phi \in C(\partial D)$ there is a unique bounded harmonic function h_ϕ on D such that $\lim_{z \to \zeta} h_\phi(z) = \phi(\zeta)$ for every $\zeta \in \partial D$ (see [5, Cor. 4.2.6]). We define L by

$$\phi \mapsto L\phi = \int h_\phi \, d\mu.$$

From the maximum principle ([5, Th. 3.6.9]) we see that $\phi \geqslant 0$ implies $L\phi \geqslant 0$. Also, $\phi \equiv 1 \Rightarrow L\phi = \mu(\bar{D})$. By the Riesz representation theorem ([5, Th. A.3.2]) there is a unique measure μ' on ∂D for which

$$L\phi = \int \phi \, d\mu',$$

for all $\phi \in C(\partial D)$ and μ' has the same mass as μ.

DEFINITION 2.1. μ' is the *sweeping out* or *balayage* of μ from $\bar{D}$, onto ∂D. If μ is not supported in $\bar{D}$, then we mean by μ' the sum of the measures obtained by sweeping out μ, restricted to the different connected components of $\overline{\mathbb{C}} \setminus \partial D$, onto ∂D respectively.

When ∂D and supp $\mu \subset \bar{D}$ are compact subsets of $\mathbb{C}$ the following holds true (see [7, Th 4.7]):

(i) $U_{\mu'}(z) \leqslant U_{\mu}(z) + c(\mu)$ for all $z \in \mathbb{C}$.

(ii) $U_{\mu'}(z) = U_{\mu}(z) + c(\mu)$ for every $z \notin D$.

The constant $c(\mu)$ is nonnegative. If D is bounded $c(\mu) = 0$ and if $\infty \in D$ then $c(\mu) = \int g_D(t, \infty)\, d\mu(t)$.

The sweeping out process is linear ($(\mu + \nu)' = \mu' + \nu'$) and continuous in the sense that $\mu_n \xrightarrow{w^*} \mu$ as $n \to \infty$ implies $\mu'_n \xrightarrow{w^*} \mu'$ as $n \to \infty$.

In the case of a point mass $\mu = \delta_z$, $z \in D$ it is well known that δ'_z is the harmonic measure on ∂D evaluated at z and in the special case where $z = \infty$ we get $\delta'_\infty = \tau$, where τ is the equilibrium measure on ∂D.

3. RESULTS

THEOREM 3.1. *Let $D \subset \mathbb{C}$ be a simply connected domain and g: $D \to \mathbb{C}$ be a locally integrable function on D. Then there exist interpolation points $\{a_{ni}\}_{i=0}^{n}$, $n = 1, 2, 3, \ldots$ such that for an arbitrary analytic function f: $D \to \mathbb{C}$ for which $|f/g|$ is bounded, we have $p_n \to f$ locally uniformly on D, where p_n is the (unique) polynomial of degree not greater than n interpolating to f at $\{a_{ni}\}_{i=0}^{n}$. The convergence is uniform, also in the following sense. Given an $\varepsilon > 0$, a locally integrable function g on D, a positive constant C and a compact set $K \subset D$, there is a natural number N such that if $z \in K$ and $n > N$, then*

$$|f(z) - p_n(z)| < \varepsilon,$$

for all analytic f on D satisfying $|f/g| < C$.

Remark 3.1. Taking g to be a constant we can guarantee convergence for all bounded analytic functions f on D.

Remark 3.2. The theorem does not tell us if the convergence is locally uniform with *geometric* degree of convergence, which is usually the case in this type of situations.

In [2] the authors examine what happens in the case of rational interpolants (and where D is not necessarily simply connected) when we let the interpolation points approach the boundary of D. The main result of that paper is the following. Assume that ∂D is bounded and $\alpha(D) > 0$ for any weak star limit point of $\{\alpha_n\}_1^\infty$ and assume also that

$$\lim_{n\to\infty} [\sup_{z\in\partial D} (U_{\alpha_n'}(z) - U_{\beta_n'}(z))] = 0, \tag{1}$$

where α_n' and β_n' denote the sweeping out onto ∂D of α_n and β_n; respectively. Then, for r_n the rational interpolant with poles at B_n interpolating to f at A_n, we have $r_n \to f$ in D, as $n \to \infty$, for any bounded, analytic function f in D. The convergence is locally uniform with geometric degree of convergence.

The following theorem states a general situation, for which we can find poles and interpolation points satisfying (1).

THEOREM 3.2. *Let $D \subset \mathbb{C}$ be a hounded regular domain. If α and β are probability measures with $\alpha(D) = 1$, $\beta(\overline{\mathbb{C}} \setminus D) = 1$ and $\alpha' = \beta'$ (where α' and β' denote the sweeping out onto ∂D of α and β, respectively), then there exist points $\{a_{ni}\}_{i=0}^n$ in D, with $\alpha_n \xrightarrow{w^*} \alpha$ and points $\{b_{ni}\}_{i=1}^n$ in $\overline{\mathbb{C}} \setminus D$ with $\beta_n \xrightarrow{w^*} \beta$ as $n \to \infty$, such that for some subsequence* (1) *holds.*

In [2, Example 2], it is shown that the condition $\alpha' = \beta'$ alone, is not sufficient to guarantee convergence for every bounded f, but the question if condition (1) (which implies $\alpha' = \beta'$) is necessary, was left open. This example shows that it is not.

EXAMPLE 3.1. Let $D = \{|z| < 1\}$. For each $n \in \mathbb{N}$, let all $b_{ni} = \infty$ for $i = 1, \ldots, n$ and $a_{ni} = 0$, $i = 0, \ldots, n-1$, $a_{nn} = a_n$, where $|a_n| \to 1$ as $n \to \infty$.

We get $\alpha_n = \frac{n}{n+1}\delta_0 + \frac{1}{n+1}\delta_{a_n}$ and for $z \in \partial D$

$$U_{\alpha_n'}(z) = \frac{n}{n+1} U_{\delta_0'}(z) + \frac{1}{n+1} U_{\delta_{\alpha_n}'}(z)$$

$$= \frac{n}{n+1} U_\tau(z) + \frac{1}{n+1} U_{\delta_{a_n}}(z) = \frac{n}{n+1} U_\tau(z) - \frac{1}{n+1} \log|z - a_n|,$$

where τ is the equilibrium measure on ∂D. Also $\beta_n' = \tau$ for all n.

Fix a $\zeta \in \partial D$. We get

$$\begin{aligned}
&\limsup_{n\to\infty} \left[\sup_{z\in\partial D} (U_{\alpha'_n}(z)-U_{\beta'_n}(z))\right] \\
&\quad \geqslant \limsup_{n\to\infty} (U_{\alpha'_n}(\zeta)-U_{\beta'_n}(\zeta)) \\
&\quad = \limsup_{n\to\infty} \left(\frac{-1}{n+1} U_\tau(\zeta)-\frac{1}{n+1}\log|\zeta-a_n|\right).
\end{aligned}$$

So, by letting a_n approach ζ fast enough, we can make this, quantity arbitrarily large or even infinite, i.e. (1) is not satisfied.

Now let $K \subset D$ be a compact set and f a bounded analytic function on D. Let γ be a positively oriented circle, with center at the origin and radius $r<1$ such that K is contained in γ. Let Γ_n be a positively oriented circle with center at the origin, radius less than one and containing γ and the point a_n inside Γ_n. For n large, a_n is outside of γ and we can let C_n be a positively oriented circle with center at a_n and with a radius so small that C_n does not intersect γ or Γ_n. For $z \in K$ and $H_n(z)=\prod_{i=0}^{n}(z-a_{ni})=z^n(z-a_n)$ we have by the Hermite interpolation formula ([8, p. 50]) that

$$\begin{aligned}
f(z)-r_n(z) &= \frac{1}{2\pi i} H_n(z)\int_{\Gamma_n} \frac{f(t)\,dt}{t^n(t-a_n)(t-z)} \\
&= \frac{1}{2\pi i} H_n(z)\int_{\gamma} \frac{f(t)\,dt}{t^n(t-a_n)(t-z)} \\
&\quad + \frac{1}{2\pi i} H_n(z)\int_{C_n} \frac{f(t)\,dt}{t^n(t-a_n)(t-z)},
\end{aligned}$$

if n is large.

For $t\in\gamma$, $z\in K$, the expression $f(t)/((t-a_n)(t-z))$ is bounded as $n\to\infty$ and the first integral is less than a constant times r^{-n}. Since $|H_n(z)| = |z-a_n|\cdot|z|^n$, $|z-a_n|<2$ for all n and $|z|/r<1$, we get that the first term tends to zero as n approaches infinity, uniformly and with geometric degree of convergence on K.

For the second term we have

$$\begin{aligned}
\frac{1}{2\pi i} H_n(z)\int_{C_n} \frac{f(t)\,dt}{t^n(t-a_n)(t-z)} &= H_n(z)\operatorname{Res}\left(\frac{f(t)}{t^n(t-a_n)(t-z)}, a_n\right) \\
&= H_n(z)\frac{f(a_n)}{(a_n)^n\,(a_n-z)} = -z^n\frac{f(a_n)}{(a_n)^n}.
\end{aligned}$$

Take an $R > r$. Since f is bounded and since $|z| < r < 1$ and $|a_n| > R$ for big n, we get $|\frac{z}{a_n}| < \frac{r}{R} < 1$ for big n, so the second term also tends to zero as n approaches infinity, uniformly and with geometric degree of convergence on K.

4. PROOFS

We start with an elementary lemma.

LEMMA 4.1. *Let μ_n, $n = 1, 2, 3, \ldots$ be probability measures, all having support in a fixed compact subset L of $\mathbb{C}$, and converging to some measure μ in the weak star sense. If $K \subset \mathbb{C}$ is a compact set with $L \cap K = \emptyset$, then $U_{\mu_n} \to U_\mu$ uniformly on K as $n \to \infty$.*

We omit the proof of this lemma, but note that it follows from the definition of U_μ and the uniform continuity of the logarithmic kernel, $\log |z - t|$, on $K \times L$.

We can now prove Theorem 3.1. The idea is based on the fact that to get convergence on a given compact subset K of D we can by Walsh's classic theorem ([8, Chapter VII]) choose the interpolation points on the boundary of K asymptotically as the equilibrium distribution of K. To get convergence on the whole of D we repeat the construction for gradually larger and larger K but making sure that we do not approach the boundary of D too fast.

Proof (Proof of Theorem 3.1). Let K_j, $j = 1, 2, 3, \ldots$ be a sequence of compact sets with $K_j \subset D$, $K_j \subset K_{j+1}$ and $\bigcup K_j = D$. Choose compacts $F_j \subset K_j$ such that $F_j \subset F_{j+1}$, $\bigcup F_f = D$ and $F_j \cap \partial K_j = \emptyset$. Let $\Gamma_j \subset D \setminus K_j$ be a simple, closed, rectifiable curve, winding once around every point in K_j. Since g is locally integrable, Γ_j can be chosen in such a way that g is integrable also on Γ_j. Given $n+1$ points $\{a_{jni}\}_{i=0}^n$ lying inside of Γ_j and p_n^j the polynomial of maximum degree n interpolating to f at $\{a_{jni}\}_{i=0}^n$ we have for z inside Γ_j that

$$R_n^j(z) := f(z) - p_n^j(z) = \frac{1}{2\pi i} \int_{\Gamma_j} \frac{H_n^j(z)}{H_n^j(t)} \cdot \frac{f(t)}{t - z} \, dt, \tag{2}$$

where $H_n^j(z) = \prod_{i=0}^n (z - a_{jni})$.

Let $\{a_{jni}\}_{i=0}^n$ be a Fekete $(n+1)$-tuple for K_j (see [5, p. 152 Definition 5.5.1]) and let $\alpha_{jn} = \frac{1}{n+1} \sum_{i=0}^n \delta_{a_{jni}}$. We have that $\alpha_{nj} = \xrightarrow{w^*} \tau_j$, as $n \to \infty$

(see [5, p. 159]), where τ_j is the equilibrium measure for K_j. Since $|f/g|$ is bounded and g is integrable on Γ_j we get by (2), that for large j

$$\|R_n^j\|_{F_j} \leqslant C_j \max\left\{\left|\frac{H_n^j(z)}{H_n^j(t)}\right| : z \in F_j,\, t \in \Gamma_j\right\} =: E_n^j, \tag{3}$$

where the constant C_j depends on g and j but not on f (the constant that bounds $|f/g|$ depends on f but is independent of j, so by letting C_j grow with j as $j \to \infty$ the inequality will hold for large j independent of f). Defining E_n^j in this way we get $\|R_n^j\|_{F_j} \leqslant E_n^j$. Raising to the power $1/n$ and taking logarithms we get

$$\frac{1}{n}\log E_n^j = \frac{1}{n}\log C_j + \frac{n+1}{n}\max\{U_{\alpha_{jn}}(t) - U_{\alpha_{jn}}(z) : z \in F_j,\, t \in \Gamma_j\}.$$

We have $\operatorname{supp}\alpha_{jn} \subset \partial K_j$ for every n ([5, p. 152]), $\partial K_j \cap F_j = \varnothing$ by assumption and $\partial K_j \cap \Gamma_j = \varnothing$ and so we have by Lemma 4.1 that

$$\limsup_{n\to\infty} \frac{1}{n}\log E_n^j \leqslant \max\{U_{\tau_j}(t) - U_{\tau_j}(z) : z \in F_j,\, t \in \Gamma_j\}. \tag{4}$$

The next step is to show that the right hand side of (4) is negative. By Frostman's theorem ([5, p. 59]), we have $U_{\tau_j}(z) = \max_{\zeta\in\mathbb{C}} U_{\tau_j}(\zeta) = I(\tau_j)$, for $z \in F_j$ and since U_{τ_j} is harmonic on $\mathbb{C}\setminus K_j$ we have, by the maximum principle and the fact that $\Gamma_j \subset \mathbb{C}\setminus K_j$ is compact, that for some $\varepsilon_j > 0$ it holds that $U_{\tau_j}(t) < I(\tau_j) - \varepsilon_j$ for all $t \in \Gamma_j$. Hence

$$\limsup_{n\to\infty} \frac{1}{n}\log E_n^j < 0,$$

so $E_n^j \to 0$ (geometrically) as $n \to \infty$.

Let $\{n_j\}$, $n_j < n_{j+1}$ be a sequence of natural numbers such that $E_n^j < 1/j$ for $n \geqslant n_j$. Given n there is a unique $j(n)$ with $n_{j(n)} \leqslant n < n_{j(n)+1}$. We choose our interpolation points $\{a_{ni}\}_{i=0}^n$ in D as $a_{ni} = a_{j(n),n,i}$. We have

$$R_n(z) := f(z) - p_n(z) = R_n^{j(n)}(z), \qquad z \in F_{j(n)},$$

so for $n > n_k$ we get

$$\|R_n\|_{F_k} \leqslant \|R_n\|_{F_{j(n)}} = \|R_n^{j(n)}\|_{F_{j(n)}} \leqslant E_n^{j(n)} < 1/j(n)$$

and since $j(n) \to \infty$ as $n \to \infty$ the first statement of the theorem is proved.

The second statement of the theorem follows from the fact that given a constant $C > 0$, we can, according to (3), find a $J \in \mathbb{N}$ such that $\|R_n^j\|_{F_j} \leqslant E_n^j$ for all f satisfying $|f/g| \leqslant C$, if $j > J$. ▮

For the proof of Theorem 3.2 we need another elementary lemma, about discretization of measures. The proof is omitted.

LEMMA 4.2. *Let μ be a probability measure on $\overline{\mathbb{C}}$. Then there are measures*

$$\mu_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{a_{ni}}, \qquad \{a_{ni}\}_{i=1}^{n} \subset \operatorname{supp} \mu,$$

such that $\mu_n \xrightarrow{w^*} \mu$ as $n \to \infty$.

The lemma is a consequence of the fact that a continuous function on the Riemann sphere is also uniformly continuous there.

We proceed by proving Theorem 3.2. The idea of the proof is similar to the proof of Theorem 3.1 in the following sense: For a measure α with $\operatorname{supp} \alpha \subset D$ the condition $\alpha' = \beta'$ together with $\operatorname{supp} \alpha_n \subset \operatorname{supp} \alpha$ as in Lemma 4.2, implies (1). In the theorem however, we require only $\alpha(D) = 1$, so to get (1) we must make sure that the points $\{a_{ni}\}_{i=0}^{n}$ which form the support of α_n, do not approach the boundary too fast when n tends to infinity. We do this by making sure that in the nth step, the support of α_n is contained in some given compact subset of D. When n tends to infinity we gradually let the support grow towards the boundary of D, but slowly enough so that (1) still will be valid.

Proof (Proof of Theorem 3.2). Using Lemma 4.2 we can choose points $\{\tilde{a}_{ni}\}_{i=0}^{n}$ in $\bar{D}$ such that for $\alpha^n := \frac{1}{n+1} \sum_{i=0}^{n} \delta_{\tilde{a}_{ni}}$ we have $\alpha^n \xrightarrow{w^*} \alpha$. Using the same Lemma again, we choose $\{b_{ni}\} \subset \overline{\mathbb{C}} \setminus D$ such that for $\beta_n := \frac{1}{n} \sum_{i=1}^{n} \delta_{b_{ni}}$, we have $\beta_n \xrightarrow{w^*} \beta$ (which implies that $\beta'_n \xrightarrow{w^*} \beta'$).

For every $j \in \mathbb{N}$, let $K_j \subset D$, $K_j \subset K_{j+1}$, be compact sets with $\bigcup K_j = D$. We write $\alpha^n = \mu^{jn} + \nu^{jn}$ where $\mu^{jn} = \alpha^n|_{K_j}$ and $\nu^{jn} = \alpha^n|_{\bar{D} \setminus K_j}$.

Fix a point a in K_1 and let $\alpha^{jn} := \mu^{jn} + c^{jn}\delta_a$, where $c^{jn} := \nu^{jn}(\overline{\mathbb{C}})$ (so we "move" all points $\tilde{a}_{ni}$ outside K_j to a fixed point a in D). Taking a subsequence and relabelling if necessary, there are measures μ^j and ν^j such that $\mu^{jn} \xrightarrow{w^*} \mu^j$ and $\nu^{nj} \xrightarrow{w^*} \nu^j$ as $n \to \infty$, where

$$\operatorname{supp} \mu^j \subset K_j, \qquad \operatorname{supp} \nu^j \subset \overline{D \setminus K_j}$$

and

$$\alpha = \mu^j + \nu^j.$$

Let $M = \max_{z,t \in \bar{D}} \log|z-t|$. We have, for $z \in \bar{D}$, that

$$U_{\nu^j}(z) = -\int \log|z-t|\, d\nu^j(t) \geqslant -M \cdot \nu^j(\bar{\mathbb{C}})$$

and since the right hand side tends to zero as j approaches infinity, we have that

$$\limsup_{j\to\infty} -U_{\nu^j}(z) < 0, \tag{5}$$

holds uniformly on $\bar{D}$. Also note that $0 \leqslant c^{jn} = \nu^{nj}(\bar{\mathbb{C}}) \leqslant \alpha^n(\overline{D\setminus K_j})$, so (by [7, Section 0, Theorem 1.3])

$$\limsup_{n\to\infty} c^{jn} \leqslant \alpha(\overline{D\setminus K_j}). \tag{6}$$

The expression $U_{\mu^{jn}} - c^{jn} \log|\cdot - a| - U_{\beta'_n}$ is upper semicontinuous on ∂D which is compact, so we can let $z^{jn} \in \partial D$ be such that

$$U_{\mu^{jn}}(z^{jn}) - c^{jn}\log|z^{jn}-a| - U_{\beta'_n}(z^{jn}) = \sup_{z\in\partial D} (U_{\mu^{jn}}(z) - c^{jn}\log|z-a| - U_{\beta'_n}(z)).$$

By taking a subsequence and relabelling if necessary, we have for some $z^j \in \partial D$ that $z^{jn} \to z^j$ as $n \to \infty$. By the principle of descent (see [6, Appendix III])

$$\limsup_{n\to\infty} -U_{\beta'_n}(z^{jn}) \leqslant -U_{\beta'}(z^j).$$

Using this, together with the continuity of U_{μ^j} on ∂D, Lemma 4.1 and (6), we see that given $j \in \mathbb{N}$, there exists an N_j, with $N_{j+1} > N_j$, such that for $n \geqslant N_j$ we have

$$U_{\mu^j}(z^{jn}) - U_{\beta'_n}(z^{jn}) < U_{\mu^j}(z^j) - U_{\beta'}(z^j) + 1/j, \tag{7}$$

$$|U_{\mu^{jn}}(z) - U_{\mu^j}(z)| < 1/j, \qquad z \in \partial D \tag{8}$$

and

$$0 \leqslant c^{jn} < \alpha(\overline{D\setminus K_j}) + 1/j. \tag{9}$$

This gives us an increasing sequence $\{N_j\} \subset \mathbb{N}$ and for every $n \geqslant N_1$ we have a unique j_n with $N_{j_n} \leqslant n < N_{j_n+1}$. Also, $\lim_{n\to\infty} j_n = \infty$.

Let $\alpha_n := \alpha^{j_n n}$. Note that α_n has the desired form as in the statement of the theorem, because since $\mu^{j_n n}$ is α_n restricted to K_{j_n} we have that for some $a_{ni} \in D$, $\alpha_n = \alpha^{j_n n} = \mu^{j_n n} + c^{j_n n}\delta_a = \frac{1}{n+1}\sum_{i=0}^{n} \delta_{a_{ni}}$ (if we have been taking subsequences and thus relabelling, $\alpha_n = \frac{1}{n'+1}\sum_{i=0}^{n'} \delta_{a_{n'i}}$ where $n' \geqslant n$). We have also that $\alpha_n \xrightarrow{w^*} \alpha$ as $n \to \infty$.

For $z \in \partial D$ we have $U_{\alpha_n'}(z) = U_{\alpha_n}(z)$ and $U_{\alpha'}(z) = U_\alpha(z)$. We use this together with (8) and (7) to get:

$$
\begin{aligned}
\sup_{z \in \partial D} (U_{\alpha_n'}(z) - U_{\beta_n'}(z)) &= \sup_{z \in \partial D} (U_{\alpha_n}(z) - U_{\beta_n'}(z)) \\
&= \sup_{z \in \partial D} (U_{\mu^{j_n n}}(z) - c^{j_n n} \log |z - a| - U_{\beta_n'}(z)) \\
&= U_{\mu^{j_n n}}(z^{j_n n}) - c^{j_n n} \log |z^{j_n n} - a| - U_{\beta_n'}(z^{j_n n}) \\
&< U_{\mu^{j_n}}(z^{j_n n}) + \frac{1}{j_n} - c^{j_n n} \log |z^{j_n n} - a| - U_{\beta_n'}(z^{j_n n}) \\
&< U_{\mu^{j_n}}(z^{j_n}) + \frac{1}{j_n} - c^{j_n n} \log |z^{j_n n} - a| - U_{\beta'}(z^{j_n}) + \frac{1}{j_n} \\
&= U_\alpha(z^{j_n}) - U_{\nu^{j_n}}(z^{j_n}) - c^{j_n n} \log |z^{j_n n} - a| - U_{\beta'}(z^{j_n n}) + \frac{2}{j_n} \\
&= -U_{\nu^{j_n}}(z^{j_n}) - c^{j_n n} \log |z^{j_n n} - a| + \frac{2}{j_n}.
\end{aligned}
$$

Using (5) and (9) we then see that

$$\limsup_{n \to \infty} [\sup_{z \in \partial D} (U_{\alpha_n'}(z) - U_{\beta_n'}(z))] \leqslant 0.$$

It remains to show that

$$\liminf_{n \to \infty} [\sup_{z \in \partial D} (U_{\alpha_n'}(z) - U_{\beta_n'}(z))] \geqslant 0. \tag{10}$$

Assume the opposite. We would, for a subsequence $\{n_k\}$ and all $z \in \partial D$, have $U_{\alpha_{n_k}'}(z) - U_{\beta_{n_k}'}(z) < c < 0$. That implies, if τ is the unit equilibrium measure on ∂D and $U_\tau = V$ on ∂D, that

$$\int V d(\alpha_{n_k}' - \beta_{n_k}') = \int (U_{\alpha_{n_k}'} - U_{\beta_{n_k}'})\, d\tau < c < 0$$

by Fubini's theorem. Letting $k \to \infty$ we get $\alpha' \neq \beta'$ which contradicts our assumption, so (10) must hold. ∎

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